ON THE
ORIGIN AND INVESTIGATION OF
PORISMS.

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BY MR. JOHN PLAYFAIR, F. R. S. EDIN.
And Professor of Mathematics in the University of Edinburgh.

FROM THE THIRD VOLUME OF THE TRANSACTIONS OF
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PART I.

1. THE restoration of the ancient books of geometry would have been impossible, without the coincidence of two circumstances, of which, though the one is purely accidental, the other is eſſentially connected with the nature of the mathematical sciences. The firſt of these circumstances is the preservation of a short abstract of those books, drawn up by PAPPUS ALEXANDRINUS, together with a series of such *lemmata*, as he judged useful to facilitate the study of them. The second is, the *necessary connection* that takes place among the objects of every mathematical work, which, by excluding whatever is arbitrary, makes it possible to determine the whole course of an investigation, when only a few points in it are known. From

the union of these circumstances, mathematics has enjoyed an advantage of which no other branch of knowledge can partake; and while the critic or the historian has only been able to lament the fate of those books of Livy and Tacitus which are lost, the geometer has had the high satisfaction to behold the works of Euclid and Apollonius reviving under his hands.

2. The first restorers of the ancient books were not, however, aware of the full extent of the work which they had undertaken. They thought it sufficient to demonſtrate the propositions, which they knew from Pappus, to have been contained in those books; but they did not follow the ancient method of investigation, and few of them appear to have had any idea of the elegant and simple analysis by which these propositions were originally discovered, and by which the Greek Geometry was peculiarly distinguished.

Among these few, Fermat and Halley are to be particularly remarked. The former, one of the greatest mathematicians of the last age, and a man in all respects of superior abilities, had very just notions of the geometrical analysis, and appears often abundantly skilful in the use of it; yet in his restoration of the *Loci Plani*, it is remarkable, that in the most difficult propositions, he lays aside the analytical method, and contents himself with giving the synthetical demonstration. The latter, among the great number and variety of his literary occupations, found time for a most attentive study of the ancient mathematicians, and was an instance of, what experience shews to be much rarer than might be expected, a man equally well acquainted with the ancient and the modern geometry, and equally disposed to do justice to the merit of both. He restored the books of Apollonius, on the problem *De Sectione Spatii*, according to the true principles of the ancient analysis.

These books, however, are but short, so that the first restoration of considerable extent that can be reckoned complete, is that of the *Loci Plani* by Dr. SIMSON, published in 1749, which, if it differs at all from the work it is intended to replace, seems to do so only by its greater excellence. This much at least is certain, that the method of the ancient geometers does not appear to greater advantage in the most entire of their writings, than in the restoration above mentioned; and that Dr. SIMSON has often sacrificed the elegance to which his own analysis would have led, in order to tread more exactly in what the *lemmata* of PAPPUS pointed out to him, as the track which APOLLONIUS had pursued.

3. There was another subject, that of *Porisms*, the most intricate and enigmatical of any thing in the ancient geometry, which was still reserved to exercise the genius of Dr. SIMSON, and to call forth that enthusiastic admiration of antiquity, and that unwearied perseverance in research, for which he was so peculiarly distinguished. A treatise in three books, which EUCLID had composed on Porisms, was lost, and all that remained concerning them was an abstract of that treatise, inserted by PAPPUS ALEXANDRINUS, in his *Mathematical Collections*, in which, had it been entire, the geometers of later times would doubtless have found wherewithal to console themselves for the loss of the original work. But unfortunately it has suffered so much from the injuries of time, that all which we can *immediately* learn from it is, that the Ancients put a high value on the propositions which they called Porisms, and regarded them as a very important part of their analysis. The Porisms of EUCLID are said to be, "Collectio artificiosissima multarum rerum quæ spectant ad analysin difficiliorum et generalium problematum.*" The curiosity, however, which is excited by this encomium is quickly disappointed; for when PAPPUS proceeds

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* Collectiones Math. lib. vii. in init.

to explain what a Porism is, he lays down two definitions of it, one of which is rejected by him as imperfect, while the other, which is stated as correct, is too vague and indefinite to convey any useful information.

These defects might nevertheless have been supplied, if the enumeration which he next gives of EUCLID's Propositions had been entire; but on account of the extreme brevity of his enunciations, and their reference to a diagram which is lost, and for the constructing of which no directions are given, they are all, except one, perfectly unintelligible. For these reasons, the fragment in question is so obscure, that even to the learning and penetration of Dr. HALLEY it seemed impossible that it could ever be explained; and he therefore concluded, after giving the Greek text with all possible correctness, and adding the Latin translation, " Hactenus Porismatum descriptio nec mihi intellecta, nec lectori profutura. Neque aliter fieri potuit, tam ob defectum schematis cujus fit mentio, quam ob omissa quædam et transposita, vel aliter vitiata in propositionis generalis expositione, unde quid sibi velit PAPPUS haud mihi datum est conjicere. His adde dictionis modum nimis contractum, ac in re difficili, qualis hæc est, minime usurpandum*."

4. It is true, however, that before this time, FERMAT had attempted to explain the nature of Porisms, and not altogether without success.† Guiding his conjectures by the definition which PAPPUS censures as imperfect, because it defined Porisms only " ab accidente," *viz*. " Porisma est quod deficit hypothesi a Theoremate Locali," he formed to himself a tolerably just notion of these propositions, and illustrated his

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* De sectione rationis, proem. p. 37.

† " Porismatum EUCLIDÆORUM renovata doctrina, et sub forma Isagoges exhibita." FERMAT Opera Varia, p. 116.

general description by examples that are in effect Porisms. But he was able to proceed no farther; and he neither proved, that his notion of a Porism was the same with EUCLID's, nor attempted to restore, or explain any one of EUCLID's propositions; much less did he suppose, that they were to be investigated by an analysis peculiar to themselves. And so imperfect indeed was this attempt, that the complete restoration of the Porisms was necessary to prove, that FERMAT had even approximated to the truth.

5. All this did not, however, deter Dr. SIMSON from turning his thoughts to the same subject, which he appears to have done very early, and long before the publication of the *Loci Plani* in 1749.

The account he gives of his progress, and of the obstacles he encountered, will be always interesting to mathematicians. "Postquam vero apud PAPPUM legeram, Porismata EUCLIDIS "collectionem fuisse artificiosissimam multarum rerum, quæ "spectant ad analysin difficiliorum et generalium problematum, "magno desiderio tenebar, aliquid de iis cognoscendi; quare "sæpius et multis variisque viis tum PAPPI propositionem "generalem, mancam et imperfectam, tum primum lib. i.

"Porisma, quod solum ex omnibus in tribus libris integrum "adhuc manet, intelligere et restituere conabar; frustra tamen, "nihil enim proficiebam. Cumque cogitationes de hac re multum mihi temporis consumpserint, atque molestæ admodum "evaserint, firmiter animum induxi hæc nunquam in posterum "investigare; præsertim cum optimus geometra HALLEIUS "spem omnem de iis intelligendis abjecisset. Unde quoties "menti occurrebant, toties eas arcebam. Postea tamen accidit, "ut improvidum et propositi immemorem invaserint, meque "detinuerint donec tandem lux quædam effulserit, quæ spem "mihi faciebat inveniendi saltem PAPPI propositionem gene-

" ralem, quam quidem multa investigatione tandem restitui. " Hæc autem paulo post una cum Porismate primo lib. i. impressa est inter Transactiones Phil. anni 1723, num, 177.*"

The propositions mentioned, as inserted in the Philosophical Transactions for 1723, are all that Dr. SIMSON published on the subject of Porisms during his life, though he continued his investigations concerning them, and succeeded in restoring a great number of EUCLID's propositions, together with their analysis. The propositions thus restored form a part of that valuable edition of the posthumous works of this geometer which the mathematical world owes to the munificence of the late Earl STANHOPE.

6. The subject of Porisms is not, however, exhausted, nor is it yet placed in so clear a light as to need no farther illustration. It yet remains to enquire into the probable origin of these propositions, that is to say, into the steps by which the ancient geometers appear to have been led to the discovery of them.

It remains also to point out the relations in which they stand to the other classes of geometrical truths; to consider the species of analysis, whether geometrical or algebraical, that belongs to them; and, if possible, to assign the reason why they have so long escaped the notice of modern mathematicians. It is to these points that the following observations are chiefly directed.

7. I begin with describing the steps that appear to have led the ancient geometers to the discovery of Porisms; and must here supply the want of express testimony by probable reasonings, such as are necessary, whenever we would trace remote discoveries to their sources, and which have more weight in mathematics than in any other of the sciences.

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* Rob. Simson, Op. reliqua, p. 319.

7. It cannot be doubted, that it has been the solution of problems, which, in all states of the mathematical sciences, has led to the discovery of most geometrical truths. The first mathematical enquiries, in particular, must have occurred in the form of questions, where something was given, and something required to be done; and by the reasonings necessary to answer these questions, or to discover the relation between the things that were given, and those that were to be found, many truths were suggested, which came afterwards to be the subjects of separate demonstration. The number of these was the greater, that the ancient geometers always undertook the solution of problems with a scrupulous and minute attention, which would scarcely suffer any of the collateral truths to escape their observation. We know from the examples which they have left us, that they never considered a problem as resolved, till they had distinguished all its varieties, and evolved separately every different case that could occur, carefully remarking whatever change might arise in the construction, from any change that was supposed to take place among the magnitudes which were given.

Now, as this cautious method of proceeding was not better calculated to avoid error, than to lay hold of every truth that was connected with the main object of enquiry, these geometers soon observed, that there were many problems which, in certain circumstances, would admit of no solution whatever: and that the general construction by which they were resolved would fail, in consequence of a particular relation being supposed among the quantities which were given.

Such problems were then said to become impossible; and it was readily perceived, that this always happened, when one of the conditions prescribed was inconsistent with the rest, so that the supposition of their being united in the same *subject*, involved a contradiction. Thus, when it was required to divide a given line, so that the rectangle under its segment, should be equal to a given space, it was evident, that if this space was greater than the square of half the given line, the thing required could not possibly be done; the two conditions, the one defining the magnitude of the line, and the other that of the rectangle under its segments, being then inconsistent with one another. Hence an infinity of beautiful propositions concerning the *maxima* and the *minima* of quantities, or the limits of the possible relations which quantities may stand in to one another.

8. Such cases as these would occur even in the solution of the simplest problems; but when geometers proceeded to the analysis of such as were more complicated, they must have remarked, that their constructions would sometimes fail, for a reason directly contrary to that which has now been assigned. Instances would be found where the lines that, by their intersection, were to determine the thing sought, instead of intersecting one another, as they did in general, or of not meeting at all, as in the above-mentioned case of impossibility, would coincide with one another entirely, and leave the question of consequence unresolved. But though this circumstance must have created considerable embarrassment to the geometers who first observed it, as being perhaps the only instance in which the language of their own science had yet appeared to them ambiguous or obscure, it would not probably be long till they found out the true interpretation to be put on it. After a little reflection, they would conclude, that since, in the general problem, the magnitude required was determined by the intersection of the two lines above mentioned, that is to say, by the points common to them both; so, in the case of their coincidence, as all their points were in common, every one of these points must afford a solution; which solutions therefore must be infinite in number; and also, though infinite in number, they

must all be related to one another, and to the things given, by certain laws, which the position of the two coinciding lines must necessarily determine.

On enquiring farther into the peculiarity in the state of the data which had produced this unexpected result, it might likewise be remarked, that the whole proceeded from one of the conditions of the problem involving another, or necessarily including it; so that they both together made in fact but one, and did not leave a sufficient number of independent conditions, to confine the problem to a single solution, or to any determinate number of solutions. It was not difficult afterwards to perceive, that these cases of problems formed very curious propositions, of an intermediate nature between problems and theorems, and that they admitted of being enunciated separately, in a manner peculiarly elegant and concise. It was to such propositions, so enunciated, that the ancient geometers gave the name of *Porisms*.

9. This deduction requires to be illustrated by examples. Suppose therefore it is proposed to resolve the following problem:

PROP. I. PROB. *Fig.* 22. *Pl.* V.

A circle ABC, a straight line DE, and a point F, being given in position, to find a point G in the straight line DE, such that GF, the line drawn from it to the given point, shall be equal to GB, the line drawn from it touching the given circle.

Suppose the point G to be found, and GB to be drawn touching the circle ABC in B; let H be the centre of the circle ABC; join HB, and let HD be perpendicular to DE; from D draw DL, touching the circle ABC in L, and join HL. Also from the centre G, with the distance GB or GF, describe the circle BKF, meeting HD in the points K and K′.

It is plain, that the lines HD and DL are given in position and in magnitude. Also, because GB touches the circle ABC, HBG is a right angle; and since G is the centre of the circle BKF, therefore HB touches the circle BKF, and consequently the square of HB, or of HL, is equal to the rectangle K′HK. But the rectangle K′HK, together with the square of DK, is equal to the square of DH, because K′K is bisected in D; therefore the squares of HL and DK are also equal to the square of DH. But the squares of HL and LD are equal to the same square of DH; wherefore the square of DK is equal to the square of DL, and the line DK to DL. But DL is given in magnitude; therefore DK is given in magnitude, and K is therefore a given point. For the same reason, K′, is a given point, and the point F being also given by hypothesis, the circle BKF is given in position. The point G therefore, the centre of the circle BKF is given, which was to be found.

Hence this construction: Having drawn HD perpendicular to DE, and DL touching the circle ABC, make DC and DK′ each equal to DL, and find G the centre of a circle described through the points K, F and K′; that is, let FK be joined, and bisected at right angles by the line MN, which meets DE in G; G will be the point required, or it will be such a point, that if GB be drawn from it, touching the circle ABC, and GF to the given point, GB and GF will be equal to one another.*

The synthetical demonstration needs not be added; but it is necessary to remark, that there are cases in which this construction fails altogether.

For, first, if the given point F be any where in the line HD, as at F′, it is evident, that MN becomes parallel to DE, and hat the point G is no where to be found, or, in other words, s at an infinite distance from D.

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* This solution of the problem was suggested to me by Professor ROBISON, and is more simple than that which I had originally given.

This is true in general; but if the given point F coincide with K, then the line MN evidently coincides with DE; so hat, agreeably to a remark already made, every point of the ine DE may be taken for G, and will satisfy the conditions of he problem; that is to say, GB will be equal to GK, wherever he point G be taken in the line DE. The same is true if F oincide with K'.

This is easily demonstrated synthetically; for if G be any point whatsoever in the line DE, from which GB is drawn ouching the circle ABC; if DK and DK' be each made equal to DL; and if a circle be described through the points B, K and K'; then, since the rectangle KHK', together with he square of DK, that is, of DL, is equal to the square of DH, that is, to the squares of DL and LH, the rectangle KHK', s equal to the square of HB, so that HB touches the circle BKK'. But BG is at right angles to HB; therefore the centre of the circle BKK' is in the line BG; and it is also in the line DE; therefore G is the centre of the circle BKK', and GB s equal to GK.

Thus we have an instance of a problem, and that too a very imple one, which is in general determinate, admitting only of one solution, but which nevertheless, in one particular case, where a certain relation takes place among the things given, becomes indefinite, and admits of innumerable solutions. The proposition which results from this case of the problem is a Porism, according to the remarks that were made above; and n effect will be found to coincide with the 66th proposition in Dr. SIMSON's restoration. It may be thus enunciated: "A circle ABC being given in position, and also a straight line DE, which does not cut the circle, a point K may be found, such that if G be any point whatever in the line ED, the straight line drawn from G to the point K, shall be equal to the straight line drawn from G, touching the circle ABC."

10. The following Porism is also derived in the same manner from the solution of a very simple problem:

PROP. II. PROB. *Fig.* 23, *Pl.* v.

A triangle ABC being given, and also a point D, to draw through D a straight line DG, such, that, perpendiculars being drawn to it from the three angles of the triangles, viz. AE, BG, CF, the sum of the two perpendiculars on the same side of DG, shall be equal to the remaining perpendiculars: or, that AE and BG together, may be equal to CF.

Suppose it done: Bisect AB in H, join CH, and draw HK perpendicular to DG.

Because AB is bisected in H, the two perpendiculars AE and BG are together double of HK: and as they are also equal to CF by hypothesis, CF must be double of HK; and CL of LH. Now, CH is given in position, and magnitude; therefore the point L is given; and the point D being also given, the line DL is given in position, which was to be found.

The construction was obvious. Bisect AB in H, join CH, and take HL equal to one third of CH; the straight line which joins the points D and L is the line required.

Now, it is plain, that while the triangle ABC remains the same, the point L also remains the same, wherever the point D may be. The point D may therefore coincide with L; and when this happens, the position of the line to be drawn is left undetermined; that is to say, any line whatever drawn through L will satisfy the conditions of the problem. Here therefore we have another indefinite case of a problem, and of consequence another Porism, which may be thus enunciated; " A triangle being given in position, a point in it may be found, such, that any straight line whatever being drawn through that point, the perpendiculars drawn to this straight line from the two angles of the triangle which are on one side of it, will be together equal to the perpendicular that is drawn to the same line from the angle on the other side of it."

11. This Porism may be made much more general; for if, instead of the angles of a triangle, we suppose ever so many points to be given in a plane, a point may be found such, that any straight line being drawn through it, the sum of all the perpendiculars that fall on that line from the given points on one side of it, is equal to the sum of the perpendiculars that fall on it from all the points on the other side of it.

Or still more generally, any number of points being given not in the same plane, a point may be found, through which if any plane be supposed to pass, the sum of all the perpendiculars which fall on that plane from the points on one side of it, is equal to the sum of all the perpendiculars that fall on the same plane from the points on the other side of it. It is unnecessary to observe, that the point to be found in these propositions, is no other than the centre of gravity of the given points; and that therefore we have here an example of a Porism very well known to the modern geometers, though not distinguished by them from other theorems.

12. The problem which follows appears to have led to the discovery of more than one Porism.

PROP. III. PROB. *Fig.* 24, *Pl.* v.

A circle ABC, and two points D and E, in a diameter of it being given, to find a point F in the circumference of the given circle, from which, if straight lines be drawn to the given points E and D, these straight lines shall have to one another the given ratio of α to β*.

Suppose the problem resolved, and that F is found, so that FE has to FD the given ratio of α to β. Produce EF any how to B, bisect the angle EFD by the line FL, and the angle DFB by the line FM.

Then, because the angle EFD is bisected by FL, FL is to LD as EF to FD, that is in a given ratio; and as ED is given, each of the segments EL, LD, is given, and also the point L.

Again, because the angle DFB is bisected by FM, EM is to MD as EF to FD, that is, in a given ratio; and therefore since ED is given, EM, MD, are also given, and likewise the point M.

But because the angle LFD is half of the angle EFD, and the angle DFM half of the angle DFB, the two angles LFD, DFM, are equal to the half of two right angles, that is, to a right angle. The angle LFM being therefore a right angle, and the points L and M being given, the point F is in the circumference of a circle described on the diameter LM, and consequently given in position.

Now, the point F is also in the circumference of the given circle ABC; it is therefore in the intersection of two given circumferences, and therefore is found.

Hence this construction: Divide ED in L, so that EL may be to LD in the given ratio of α to β; and produce ED also to M, so that EM may be to MD in the same given ratio of α to β. Bisect LM in N, and from the centre N, with the distance NL, describe the semicircle LFM, and the point F in which it intersects the circle ABC, is the point required, or that from which FE and FD are to be drawn.

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* The ratio of α to β is supposed that of a greater to a less.

The synthetical demonstration follows so readily from the preceding analysis, that it is not necessary to be added.

It must, however, be remarked, that the construction fails when the circle LFM falls either wholly without, or wholly within the circle ABC, so that the circumferences do not intersect; and in these cases the solution is impossible. It is plain also, that in another case the construction will fail, *viz.* when it so happens that the circumference LFM wholly coincides with the circumference ABC. In this case, it is farther evident, that every point in the circumference ABC will answer the conditions of the problem, which therefore admits of innumerable solutions, and may, as in the foregoing instances, be converted into a Porism.

13. We are therefore to enquire, in what circumstances the point L may coincide with the point A, and the point M with the point C, and of consequence the circumference LFM with the circumference ABC.

On the supposition that they coincide, EA is to AD, and also EC to CD, as α to β; and therefore EA is to EC as AD to CD, or, by conversion, EA to AC as AD to the excess of CD above AD, or to twice DO, O being the centre of the circle ABC. Therefore also, EA is to AO, or the half of AC, as AD to DO, and EA together with AO to AO, as AD together with DO to DO; that is, EO to AO as AO to DO, and so the rectangle EO·OD equal to the square of AO.

Hence, if the situation of the points E and D (*Fig.* 25, *Pl.* v.) in respect of the circle ABC, be such, that the rectangle EO·OD is equal to the square of AO, the semidiameter of the circle; and if, at the same time, the given ratio of α to β be the same with that of EA to AD, or of EC to CD, the problem admits of innumerable solutions; and as it is manifest, that if the circle ABC, and one of the points D or E be given the other point, and also the ratio which is required to render the problem indefinite, may be found, therefore we have this Porism: "A circle ABC being given, and alſo a point D, a point E may be found, such that the two lines inflected from these points to any point whatever in the circumference ABC, shall have to one another a given ratio, which ratio is also to be ſound."

This Porism is the second in the treatise *De Porismatibus* where Dr. SIMSON gives it, not as one of EUCLID's propositions, but as an illustration of his own definition. It answers equally well for the purpose I have here in view, the explaining the origin of Porisms; and I have been the more willing to introduce it, that it has afforded me an opportunity of giving what seems to be the simplest investigation of the second proposition in the second book of the *Loci Plani*, by proving, as has been done above, that on the hypothesis of that proposition LFM (*Fig.* 24, *Pl.* v.) is a right angle, and L and M given points.

14. Hence also an example of the derivation of Porisms from one another. For the circle ABC, and the points E and D, remaining as in the last construction, (*Fig.* 25, *Pl.* v.) if through D we draw any line whatever HDB, meeting the circle in B and H, and if the lines EB, EH be also drawn, these lines will cut off equal circumferences BF and HG. Let FC be drawn, and it is plain from the foregoing analyſis, that the angles DFC, CFB are equal. Therefore if OG, OB be drawn, the angles BOC, COG are equal, and conſequently the angles DOB, DOG. In the same manner, by joining AB, the angle DBE being bisected by BA, it is evident, that the angle AOF is equal to the angle AOH, and therefore the angle FOB to the angle HOG, that is, the arch FB to the arch HG.

Now, it is plain, that if the circle ABC, and one of the points D or E be given, the other point may be found; therefore we have this porism, which appears to have been the last but one in the third book of **Euclid**'s Porisms *.

" A point being given, either without or within a circle given in position, if there be drawn, any how through that point, a line cutting the circle in two points ; another point may be found, such, that if two lines be drawn from it to the points, in which the line already drawn cuts the circle, these two lines will cut off from the circle equal circumferences."

There are other Porisms that may be deduced from the same original problem, (§ 12.) all connected, as many remarkable properties of the circle are, with the *harmonical division* of the diameter.

15. The preceding proposition also affords a good illustration of the general remark that was made above, concerning the conditions of a problem being involved in one another, in the Porismatic, or indefinite case. Thus, several independent conditions are here laid down, by help of which the problem is to be resolved : Two points D and E are given, (*Fig.* 24, *Pl.* V.) from which two lines are to be inflected, and a circumference ABC, in which these lines are to meet, as also a ratio, which they are to have to one another. † Now, these conditions are all independent of each other, so that any one of them may be changed, without any change whatever in the rest. This at least is true in general ; but nevertheless in any one case, *viz.* when the given points are so related to one another, that the rectangle under their distances from the centre, is equal to the square of the radius of the circle, it follows from the foregoing analysis, that the ratio which the inflected lines are to have to one another, is no longer a matter of choice, but is a necessary consequence

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* Simpson De Porismatibus, Prop. 53.

† The given points, and the centre of the given circle, are understood, throughout, to be in the same straight line.

of this diſpoſition of the points. For if any other ratio were now aſſigned than that of AO to OD, or, which is the ſame, of EA to AD, it would eaſily be ſhewn, that no lines having that ratio could be inflected from the points E and D, to any point in the circle ABC. Two of the conditions are therefore reduced into one; and hence it is that the problem is indefinite.

16. From this account of the origin of Poriſms, it follows, that a Poriſm may be defined, *A propoſition affirming the poſſibility of finding ſuch conditions as will render a certain problem indeterminate, or capable of innumerable ſolutions.*

To this definition, the different characters which PAPPUS has given will apply without difficulty. The propoſitions deſcribed in it like thoſe which he mentions, are, ſtrictly ſpeaking, neither theorems nor problems, but of an intermediate nature between both; for they neither ſimply enunciate a truth to be demonſtrated, nor propoſe a queſtion to be ſolved: but are affirmations of a truth, in which the determination of an unknown quantity is involved. In as far therefore as they aſſert, that a certain problem may become indeterminate, they are of the nature of theorems; and in as far as they ſeek to diſcover the conditions by which that is brought about, they are of the nature of problems.

17. In the preceding definition alſo, and the inſtances from which it is deduced, we may trace that imperfect deſcription of Poriſms which PAPPUS aſcribes to the later geometers, *viz.* "Poriſma eſt quod deficit hypotheſi a theoremate locali." Now, to underſtand this, it muſt be obſerved, that if we take the converſe of one of the propoſitions called *Loci*, and make the conſtruction of the figure a part of the hypotheſis, we have what was called by the ancients a Local Theorem. And again, if, in enunciating this theorem, that part of the hypotheſis which

contains the construction be suppressed, the proposition arising thence will be a Porism; for it will enunciate a truth, and will also require, to the full understanding and investigation of that truth, that something should be found, *viz.* the circumstance in the construction, supposed to be omitted.

Thus when we say; If from two given points E and D (*Fig.* 25, *Pl.* v.), two lines EF and FD are inflected to a third point F, so as to be to one another in a given ratio, the point F is in the circumference of a circle given in position: we have a *Locus*.

But when conversely it is said; If a circle ABC, of which the centre is O, be given in position, as also a point E, and if D be taken in the line EO, so that the rectangle EO, OD be equal to the square of AO, the semidiameter of the circle; and if from E and D, the lines EF and DF be inflected to any point whatever in the circumference ABC; the ratio of EF to DF will be a given ratio, and the same with that of EA to AD; we have a local theorem.

And, lastly, when it is said; If a circle ABC be given in position, and also a point E, a point D may be found, such, that if the two lines EF and FD be inflected from E and D to any point whatever F, in the circumference, these lines shall have a given ratio to one another; the proposition becomes a Porism, and is the same that has been just investigated.

Here it is evident, that the local theorem is changed into a Porism, by leaving out what relates to the determination of the point D, and of the given ratio. But though all propositions formed in this way, from the conversion of *Loci*, be Porisms, yet all Porisms are not formed from the conversion of *Loci*. The first and second of the preceding, for instance, cannot by conversion be changed into *Loci*; and therefore the definition which describes all Porisms as being so convertible, is not sufficiently comprehensive. FERMAT's idea of Porisms, as has been already observed, was founded wholly on this definition, and therefore could not fail to be imperfect.

18. It appears, therefore, that the definition of Porisms given above (§ 16.) agrees with PAPPUS's idea of these propositions, as far at least as can be collected from the imperfect fragments which contain his general description of them. It agrees also with Dr. SIMSON's definition, which is this [*]: "Porisma est propositio in qua proponitur demonstrare rem aliquam, vel plures datas esse, cui, vel quibus, ut et cuilibet ex rebus innumeris, non quidem datis, sed quæ ad eà quæ data sunt eandem habent relationem, convenire ostendendum est affectionem quandam communem in propositione descriptam."

It cannot be denied, that there is a considerable degree of obscurity in this definition [†], notwithstanding of which it is certain, that every proposition to which it applies must contain a *problematical* part, *viz.* "in qua proponitur demonstrare rem aliquam, vel plures datas esse," and also a *theoretical* part, which contains the property, or *communis affectio*, affirmed of certain things which have been previously described.

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[*] SIMSON's Opera Reliqua, p. 323.

[†] The following translation will perhaps be found to remedy some of the obscurity complained of.

"A PORISM is a proposition, in which it is proposed to demonstrate, that one or more things are given, between which and every one of innumerable other things, not given, but assumed according to a given law, a certain relation, described in the proposition, is to be shewn to take place."

It may be proper to remark, that there is an ambiguity in the word *given*, as used here and on many other occasions, where it denotes indifferently things that are both *determinate* and *known*, and things that, though *determinate*, are *unknown*, provided they can be found. This holds as to the first application of the term in the above definition; from which however no inconveniency arises, when the reader is apprised of it. In the course of this paper, I have endeavoured, as much as possible, to avoid the like ambiguity.

It is also evident, that the subject of every such proposition, is the relation between magnitudes of three different kinds; determinate magnitudes which are given; determinate magnitudes which are to be found; and indeterminate magnitudes which, though unlimited in number, are connected with the others by some common property. Now, these are exactly the conditions contained in the definitions that have been given here.

19. To confirm the truth of this theory of the origin of Porisms, or at least the justness of the notions founded on it, I must add a quotation from an essay on the same subject, by a member of this society, the extent and correctness of whose views make every coincidence with his opinions peculiarly flattering. In a paper read several years ago before the Philosophical Society, Professor Dugald Stewart defined a Porism to be "A proposition affirming the possibility of finding one or more of the conditions of an indeterminate theorem." Where, by an indeterminate theorem, as he had previously explained it, is meant one which expresses a relation between certain quantities that are determinate and certain others that are indeterminate, both in magnitude and in number. The near agreement of this with the definition and explanations which have been given above, is too obvious to require to be pointed out; and I have only to observe, that it was not long after the publication of Simson's posthumous works, when, being both of us occupied in speculations concerning Porisms, we were led separately to the conclusions which I have now stated. *

* In an enquiry into the origin of Porisms, the etymology of the term ought not to be forgotten. The question indeed is not about the derivation of the word Πορισμα, for concerning that there is no doubt; but about the reason why this term was applied to the class of propositions above described. Two opinions may be formed on this subject, and each of them with considerable probability: 1mo. One of the significa-

20. We might next proceed to consider the particular Porisms which Dr. **Simson** has restored, and to shew, that every one

tions of ϖοριζω, is to *acquire or obtain*; and hence Πορισμα, *the thing obtained or gained.*

Accordingly, **Scapula** says, *Est vox a geometris desumpta qui theorema aliquid ex demonstrativo syllogismo necessario sequens inferentes, illud quasi lucrari dicuntur, quod non ex professo quidem theorematis hujus instituta sit demonstratio, sed tamen ex demonstratis recte sequatur.* In this sense **Euclid** uses the word in his Elements of Geometry, where he calls the corollaries of his proposition, *Porismata.* This circumstance creates a presumption, that when the word was applied to a particular class of propositions, it was meant, in both cases, to convey nearly the same idea, as it is not at all probable, that so correct a writer as **Euclid**, and so scrupulous in his use of words, should employ the same term to express two ideas which are perfectly different. May we not therefore conjecture, that these propositions got the name of Porisms, entirely with a reference to their origin. According to the idea explained above, they would in general occur to mathematicians when engaged in the solution of the more difficult problems, and would arise from those particular cases, where one of the conditions of the data involved in it some one of the rest. Thus a particular kind of theorem would be obtained, following as a corollary from the solution of the problem: and to this theorem the term Ποϱισμα might be very properly applied, since, in the words of **Scapula**, already quoted, *Non ex professo theorematis hujus instituta sit demonstratio, sed tamen ex demonstratis recte sequatur.*

2do. But though this interpretation agrees so well with the supposed origin of Porisms, it is not free from difficulty. The verb ϖοϱιζω has another signification, *to find out, to discover, to devise*; and is used in this sense by **Pappus**, when he says that the propositions called Porisms, afford great delight, τοις δυναμενοις οϱαν και ϖοϱιζμα, *to those who are able to understand and* INVESTIGATE. Hence comes ϖοϱισμος, *the act of finding out or discovering*, and from ϖοϱισμος, in this sense, the same author evidently considers Ποϱισμα as being derived. His words are, Εφασαν δε (ὁι αϱχαιοι) Ποϱισμα ειναιτο ϖϱοτεινομανον εις Πο ισμον αυτȣ ϖϱοτεινομανȣ, *the Ancients said, that a Porism is something proposed for the* FINDING OUT, *or* DISCOVERING *of the very thing proposed.* It seems singular, however, that Porisms should have taken their name from a circumstance common to them with so many other geometrical truths; and if this was really the case, it must have been on account of the enigmatical form of their enunciations, which required, that in the analysis of these propositions, a sort of double discovery should be made, not only of the Truth, but also of the Meaning *of the very thing which was proposed.* They may therefore have been called *Porismata,* or *investigations*, by way of eminence.

of them is the indeterminate case of some problem. But of this it is so easy for any one, who has attended to the preceding remarks, to satisfy himself, by barely examining the enunciations of those propositions, that the detail into which it would lead seems to be unnecessary. I shall therefore go on to make some observations on that kind of analysis which is particularly adapted to the investigation of Porisms.

If the idea which we have given of these propositions be just, it follows, that they are always to be discovered by considering the cases in which the construction of a problem fails in consequence of the lines which, by their intersection, or the points which, by their position, were to determine the magnitude required, happening to coincide with one another. A Porism may therefore be deduced from the problem it belongs to, in the same manner that the propositions concerning the *maxima* and *minima* of quantities are deduced from the problems of which they form the limitations; and such no doubt is the most natural and most obvious analysis of which this class of propositions will admit.

It is not, however, the only one that they will admit of; and there are good reasons for wishing to be provided with another, by means of which, a Porism that is any how suspected to exist, may be found out, independently of the general solution of the problem to which it belongs. Of these reasons, one is, that the Porism may perhaps admit of being investigated more easily than the general problem admits of being resolved; and another, is that the former, in almost every case, helps to discover the simplest and most elegant solution that can be given of the latter.

The truth of this last observation has been already exemplified in two of the preceding problems, where the porismatic case, by determining the point K in the first, and L in the second of them, became necessary to the general solution. In more difficult problems, the same will be found to hold still more remarkably, and this is evidently what PAPPUS had in view, when, in a passage already quoted he called Porisms, "Collectio artificiosessima multarum rerum quæ spectant ad analysin difficiliorum et generalium problematum." On this account, it is desirable to have a method of investigating Porisms, which does not require, that we should have previously resolved the problems they are connected with, and which may always serve to determine, whether to any given problem there be attached a Porism, or not. Dr. SIMSON's analysis may be considered as answering to this description; for as that geometer did not regard these propositions at all in the light that is done here, nor in relation to their origin, an independent analysis of this kind, was the only one that could occur to him; and he has accordingly given one which is extremely ingenious, and by no means easy to be invented, but which he uses with great skilfulness and dexterity throughout the whole of his Restoration.

It is not easy to ascertain whether this be the precise method used by the ancients. Dr. SIMSON had here nothing to direct him but his genius, and has the full merit of the first inventor. It seems probable, however, that there is at least a great affinity between the methods, since the *lemmata* given by PAPPUS as necessary to EUCLID's demonstrations, are subservient also to those of our modern geometer.

21. I shall employ the same sort of analysis in the Porisms that follow, at least till we come to treat of them algebraically, where a method of investigating these propositions will present itself, which is perhaps more simple and direct than any other. The following Porism is the first of EUCLID's, and the first also that was restored. It is given here to exemplify the advantage which, in investigations of this kind, may be derived from

employing the law of extremity in its utmost extent, and pursuing Porisms to those extreme cases, where the indeterminate magnitudes increase *ad infinitum*; into which state Dr. SIMSON probably did not think it safe to follow them, and was thereby deprived of no inconsiderable help toward the simplifying of his constructions. If therefore it can be shewn, that this help may be obtained without any sacrifice of geometrical accuracy it will be some improvement in this branch of the analysis.

The Porism just mentioned may be considered as having occurred in the solution of a problem. Suppose it were required; two points A and B, (*Fig.* 26, *Pl.* v.) and also three straight lines DE, FK, KL, being given in position, together with two points H and M, in two of these lines, to inflect from A and B to a point in the third line, two lines that shall cut off from KF and KL two segments, adjacent to the given points H and M, having to one another the given ratio of α to β.

Now in order to find whether there be any Porism connected with this problem, suppose that there is, and that the following proposition is true.

PROP. IV. PORISM. *Fig.* 26, *Pl.* v.

22. Two points A and B, and two straight lines DE and FK, being given in position, and also a point H in one of them, a line LK may be found and also a point in it M, both given in position, such, that AE and BE, inflected from the points A and B to any point whatsoever of the line DE, shall cut off from the other lines FK and LK, segments, HG and MN, adjacent to the given points H and M, having to one another the given ratio of α to β.

First, let AE′, BE′ be inflected to the point E′, so that AE′ may be parallel to FK, then shall E′B be parallel to KL, the line to be found. For if it be not parallel to KL, the point of their intersection must be at a finite distance from the point M, and therefore making as β to α, so this distance to a fourth proportional, the distance from H, at which AE′ intersects FK, will be equal to that fourth proportional. But AE′ does not intersect FK, for they are parallel by construction; therefore BE′ cannot intersect KL; KL is therefore parallel to BE′, a line given in position.

Again, let AE″, BE″ be inflected to E″, so that AE″ may pass through the given point H; then it is plain, that BE″ must pass through the point to be found M; for if not, it may be demonstrated, just as has been done above, that AE″ does not pass through H, contrary to the supposition. The point to be found is therefore in the line E″B, which is given in position.

Now, if from E there be drawn EP parallel to AE′, and ES parallel to BE′, BS is to SE as BL to LN and AP to PE as AF to FG; wherefore the ratio of FG to LN is compounded of the ratios of AF to BL, PE to SE, and BS to AP. But the ratio of PE to SE is the same with that of AE′ to BE′, and the ratio of BS to AP is the same with that of DB to DA, because DB is to BS as DE′ to E′E. Or as DA to AP. Therefore the ratio of FG to LN is compounded of the ratios of AF to BL, AE′ to BE′, and DB to DA.

In like manner, because E″ is a point in the line DE, and AE″, BE″ are inflected to it, the ratio of FH to LM, is compounded of the same ratios of AF to BL, AE′ to BE′ and DB to DA; and therefore the ratio of FH to LM is the same with that of FG to NL, and the same consequently with that of HG to MN. But the ratio of HG to MN is given, being by supposition that of α to β; the ratio of FH to LM is therefore also given, and FH being given, LM is given in magnitude. Now, LM is parallel to BE′, a line given in position; therefore M is in a line QM parallel to AB, and given in position. But the point M is also in another line BE″ given in position; therefore the point M, and also the line KLM drawn through it parallel to BE′, are given in position, which were to be found.

The construction is thus: From A draw AE′ parallel to FK, meeting DE in E′; join BE′, and take in it BQ, so that as α to β so HF to BQ, and through Q draw QM parallel to AB. Let HA be drawn, and produced till it meet DE in E″, and let BE″ be drawn meeting QM in M. Through M draw KML parallel to BE′; then is KML the line, and M the point, which were to be found.

It is plain, that there are two lines which will answer the conditions of the Porism; for if in QB, produced on the other side of B there be taken Bq equal to BQ; and if qm be drawn parallel to AB, intersecting MB in m; and if $m\lambda$ be drawn parallel to BQ, the part mn, cut off by EB produced, will be equal to MN, and have to HG the ratio required.

It is plain also, that whatever be the ratio of α to β, and whatever be the magnitude of FH, if the other things given remain the same, the lines found will be all parallel to BE′. But if the ratio of α to β remain the same likewise, and if only the point H vary, the position of KL will remain the same, and the point M will vary.

23. This construction, from which, and the foregoing analysis, the synthetical demonstration follows readily, will be found to be more simple than Dr. SIMSON's, owing entirely to the use that has been made of the *law of continuity* in the two extreme cases, where, according to the language of the modern analysis, HG becomes infinite, in the one, and equal to nothing, in the other. Had it been affirmed, agreeably to that same language, that in the first of those cases, because of the constant ratio of HG to MN, these lines must both become infinite at the same time, and in the second, that for the same reason

they must both vanish at the same time, we might have been accused of departing from the strict form of reasoning employed in the ancient geometry. But when the thing is stated as above, and it is proved, that when AE′ does not meet KF, it is impossible for BE′ to meet ML; and again, that when AE″ passes through H, it is impossible for BE″ not to pass through M, the air of paradox is entirely removed, and the tracing of the law of continuity is rendered perfectly consistent with the utmost severity of geometrical demonstration.

Dr. SIMSON has applied this Porism very ingeniously to the solution of the same problem from which it is here supposed to be derived*; and it is worthy of remark, that supposing the points A and B, and the lines DE and FK to be as in the figure of this Porism, if the third of the given lines be not parallel to BE′, that problem can always be resolved, and admits of two solutions; but if it be parallel to BE′, the problem either becomes impossible, or a Porism; that is, it either admits of no solution, or of an infinite number.

We shall soon have occasion to extend the same observation to other Porisms.

Another general remark which I have to make on the analysis of Porisms is, that it frequently happens as in the last example, that the magnitudes required may all, or a part of them be found by considering the extreme cases; but for the discovery of the relation between them, and the indefinite magnitudes or *res innumeræ*, we must have recourse to the hypothesis of the Porism in its most general, or indefinite form, and must endeavour so to conduct the reasoning that the indefinite magnitudes shall at length wholly disappear, and leave a proposition containing only a relation of determinate magnitudes to one another. Now in order to accomplish this, Dr. SIMSON frequently employs two statements of the general hypothesis, which he compares together; as for instance, in his analysis of the last Porism, he assumes, not only E, any point whatsoever in the line DE, but also another point O, any whatsoever in the same line, to both of which he supposes lines to be inflected from the given points A and B. This double statement, however, cannot be made without rendering the investigation long and complicated; and therefore it may be of use to remark, that it is never necessary, but may always be avoided by an appeal to simpler Porisms, or to *Loci*, or to the propositions of the *data*. I shall give the following Porism as an example where this is done with some difficulty, but with considerable advantage, in regard to the simplicity and shortness of the investigation.

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* Opera Reliqua, de Porismatibus, prop. 25.

PROP. V. PORISM. *Fig.* 27, *Pl.* v.

24. Let there be three straight lines AB, AC, CB given in position, and from any point whatsoever in one of them, as D, let perpendiculars be drawn to the other two, as DF, DE; a point G may be found, such, that if GD be drawn from it to the point D, the square of that line shall have a given ratio to the sum of the squares of the perpendiculars DF and DE, which ratio is to be found.

Draw from A and B the lines AH, BK at right angles to BC and CA, and divide AB in L so that AL may be to LB in the given ratio of the square AH to the square of BK, or which is the same, of the square of AC to the square of CB. The point L is therefore given; and if N be taken so as to have to AL the same ratio that AB^2 has to AH^2, N will be the given magnitude. Also since $AH^2 : BK^2 :: AL : LB$, and $AH^2 : AB^2 :: AL : N$, *ex equo* $BK^2 : AB^2 :: LB : N$. From

L draw LO, LM perpendicular to AC, CB; LO and LM are given in magnitude.

Now, because $AB^2 : BK^2 :: AD^2 : DF^2$, $N : LB :: AD^2 : DF^3$, so that $DF^2 = \frac{LB}{N} \cdot AD^2$, and for the same reason, $DE^2 = \frac{AL}{N} \cdot BD^2$. But *(Loci Plani, Append. Lem.* 1.*)* $\frac{LB}{N} \cdot AD^2 + \frac{AL}{N} \cdot BD^2 = \frac{LB}{N} \cdot AL^2 + \frac{AL}{N} \cdot BL^2 + \frac{AB}{N} \cdot DL^2$; that is, $DE^2 + DF^2 = LO^2 + LM^2 + \frac{AB}{N} \cdot DL^2$.

Join LG; then by hypothesis, $LO^2 + LM^2$ has to LG^2 the same ratio which $DF^2 + DE^2$ has to DG^2; and if this ratio be that of R to N, $LO^2 + LM^2 = \frac{R}{N} \cdot LG^2$; and therefore $DE^2 + DF^2 = \frac{R}{N} \cdot LG^2 + \frac{AB}{N} \cdot DL^2$. But $DE^2 + DF^2 = \frac{R}{N} \cdot DG^2$; therefore $\frac{R}{N} \cdot LG^2 + \frac{AB}{N} \cdot DL^2 = \frac{R}{N} \cdot DG^2$, and $\frac{AB}{N} \cdot DL^2 = \frac{R}{N}(DG^2 - LG^2)$.

The excess of the square of DG above the square of LG, has therefore a constant ratio to the square of DL, *viz.* that of AB to R. The angle DLG is therefore a right angle and the ratio of AB to R, the ratio of equality, otherwise LD would be given in magnitude, which is contrary to the supposition. The line LG is therefore given in position; and since R is to N, that is, AB to N, as the squares of LO and LM to the square of LG, therefore the square of LG, and consequently the line LG, is given in magnitude. The point G is therefore given, and also the ratio of the squares of DE and DF to the square of DG, which is the same with that of AB to N.

Hence this construction: Divide AB in L, so that AL may be to LB as the square of AH to the square of BK, and make as the square of AH to the square of AB, so AL to N; and, lastly, having drawn from L upon AC and CB the perpendiculars LO and LM, make LG perpendicular to AB, and such, that as AB to N, so the sum of the squares of LO and LM to the square of LG; G will be the point required, and the given ratio, which the squares on DF and DE have to the square on DG, will be that of AB to N.

This is the construction which follows most directly from the analysis; but it may be rendered more simple. For since $AH^2 : AB^2 :: AL : N$, and $BK^2 : AB^2 :: BL : N$; therefore $AH + BK^2 : AB^2 :: AB : N$. Likewise, if AG, BG be joined, $AB : N :: AH^2 : AG^2$, and $AB : N :: BK^2 : BG^2$; wherefore $AB : N :: AH^2 + BK^2 : AG^2 + BG^2$, that is, $AH^2 + BK^2 : AB^2 :: AH^2 + BK^2 : AG^2 + BG^2$, and $AG^2 + GB^2 = AB^2$. The angle AGB is therefore a right angle, and $AL : LG :: LG : LB$. If therefore AB be divided in L, as in the preceding construction; and if LG, a mean proportional between AL and LB, be placed at right angles to AB, G will be the point required.

Cor. It is evident from the construction, that if at the points A and B we suppose weights to be placed that are as the squares of the sines of the angles CAB, CBA, L will be the centre of gravity of these weights. For AL is to LB as AC^2 to CB^2, or inversely as the squares of the sines of the angles at A and B.

25. Now, the step in this analysis by which a second introduction of the general hypothesis is avoided, is that in which the angle GLD is concluded to be a right angle. This conclusion follows from the excess of the square of DG above the square of GL, having a given ratio to the square of LD, at the same time that LD is of no determinate magnitude. For if possible, let GLD be obtuse, (*Fig.* 28, *Pl.* v.) and let the perpendicular from G upon AB meet AB in V, which point V

is therefore given. And since the excess of the square of DG above the square of LG is equal to the square of LD, together with twice the rectangle DLV, therefore by the supposition, the square of LD, together with twice the rectangle DLV must have a given ratio to the square of LD; the ratio of the rectangle DLV to the square of LD, that is, of VL to LD, is therefore given, so that VL being given in magnitude, LD is also given. But this is contrary to the supposition, for LD is indefinite by hypothesis; and therefore GLD cannot be obtuse, nor any other than a right angle.

The same conclusion that is here drawn immediately from the indetermination of LD, would be deduced, according to Dr. SIMSON's method, by assuming another point D′, any how, and from the supposition, that the excess of GD'^2 above GL^2, was to LD'^2 in the same ratio that the excess of GD^2 above GL^2 is to LD^2, it would follow without much difficulty, that GLD must be a right angle, and the given ratio, a ratio of equality. The method followed above is shorter and less intricate than this last, and has, I think, the advantage of discovering more plainly the *spirit* of the analysis, and the effect which the indefinite nature of the quantities, supposed indeterminate in the Porism, has in ascertaining the relation, that must subsist between the magnitudes that are given, and those that are to be found.

26. This Porism may be extended to any number of lines whatsoever, and may be thus enunciated: "Let there be any number of straight lines given in position, and from any point in one of them, let perpendiculars be drawn to all the rest, a point may be found such that the square of the line joining it and the point from which the perpendiculars are drawn, shall have to the sum of the squares of these perpendiculars a given ratio, which ratio is also to be found."

The analysis of the Porism, when thus generalized, is too long to be given here *. We must not, however, omit to take notice, that the point L, where the perpendicular from the point to be found meets the line, from which the perpendiculars are drawn to the rest, is in all cases determined by the rule suggested in the corollary. (§ 24.) For if at the points in which the said line is intersected by the others, there be placed weights proportional to the squares of the sines of the angles of intersection, L will be the centre of gravity of these weights.

27. These Porisms facilitate the solution of the general problems from which they are derived. For if it were proposed, three straight lines AB, AC, BC being given in position, and also a point R, (*Fig.* 27, *Pl.* v.), to find a point D in one of the given lines AB, such, that the sum of the squares of the perpendiculars drawn from D to the other two lines, should have a given ratio to the square of DR, it is plain, that the finding of the point G in the Porism, would render the construction easy. For the squares of RD and GD, having each given ratios to the sum of the squares of the perpendiculars drawn from D, have a given ratio to one another.

The ratio of the lines, RD and GD themselves, is therefore given, and the points R and G being given, D is the circumference of a circle given in position; and it is also in the straight line AB given in position; therefore it is given. The same holds, whatever be the number of lines given in position.

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* This Porism, in the case considered above, *viz.* when there are three straight lines given in position, was communicated to me several years ago, without any analysis or demonstration, by Dr. TRAIL, Prebendary of Lisburn in Ireland, who told me also, that he had met with it among some of Dr. SIMSON's papers, which had been put into his hands, at the time when the Posthumous Works of that Geometer were preparing for the press. The application of it to the second of the problems, (§ 27.) was also suggested by Dr. TRAIL.

The same Porisms assist also in the solution of another problem. For if it were proposed to find D, so that the sum of the squares of the perpendiculars drawn from it to AC, and CB, should be equal to a given square, this would be done by finding G; and then because the sum of the squares of the perpendiculars is given, and has a given ratio to the square of DG, DG will be given, and consequently the point D. This is also true, whatever be the number of the lines.

28. The connection of the Porisms with the impossible cases of these problems, is abundantly evident; the point L being that from which, if perpendiculars be drawn to AC and CB, the sum of their squares is the least possible. For since (*Fig.* 27, *Pl.* v.) $DF^2 + DE^2 : DG^2 :: LO^2 + LM^2 : LG^2$, and since LG is less than DG, $LO^2 + LM^2$ must be less than $DF^2 + DE^2$.

Hence also a point Q may be found, from which, if perpendiculars be drawn to the sides of the triangle ABC, the sum of the squares of these perpendiculars is less than the sum of the squares of the perpendiculars drawn to the sides of the triangles from any other point.

For if *ab* (*Fig.* 29, *Pl.* vi.) be any line drawn parallel to AB, and if it be divided in λ, so that *a*λ may be to λ*b* in the duplicate ratio of *a*C to C*b*, or of AC to CB, then of all the points in the line *ab*, λ is that from which, if perpendiculars be drawn to the lines AC, CB, the sum of their squares is the least possible. But since *a*λ is to λ*b* as the square of AC to the square of BC, that is, as AL to LB, therefore the *locus* of λ is the straight line LC, joining the given points L and C.

The point to be found therefore, or that from which perpendiculars being drawn to the sides of the figure, the sum of their squares is the least possible, is in the straight line LC.

For let q be any point on either side of LC, and let the line ab be drawn through q parallel to AB, meeting LC in λ, then the sum of the squares of the perpendiculars from q upon AC, CB, is greater than the sum of the squares of the perpendiculars from λ upon the same lines. Therefore adding the square of the perpendicular from q, or λ, on AB, to both, the sum of the squares of the perpendiculars from q, will be greater than the sum of the squares of the perpendiculars from λ. The point, therefore, which makes the sum of the squares of the perpendiculars drawn from it, to the sides of the triangle ABC, a *minimum*, is not on either side of the line LC; it is therefore in the line LC.

For the same reason, if AC be divided in L′, so that AL′ is to L′C as the square of AB to the square of BC, and if BL′ be joined, the point to be found is in BL′. It is therefore in the point Q, where the lines CL and BL′ intersect one another.

The point Q, in any other figure, may be found nearly in the same manner. Let ABCD, for instance, (*Fig.* 30, *Pl.* vi.) be a quadrilateral figure; let the opposite sides, AB and DC be produced till they meet in E, and let ab be drawn parallel to AB, meeting CE in e, and let λ be the point in the line ab from which perpendiculars are drawn to the three lines BC, CD, DA, so that the sum of their squares is less, than if they were drawn from any other point, in the same line; then if weights be placed at b, a and e, proportional to the squares of the sines of the angles Cba, baD, aeD, λ is the centre of gravity of these weights. (§ 26.) Now, these weights having given ratios to one another, the *locus* of the point λ, from the known properties of the centre of gravity, is a straight line Lλ, given in position. The point to be found is, therefore, in that line. For the same reason, it is in another straight line L′λ′ also given in position; and therefore it is in Q, the point of their intersection.

There are many other remarkable properties of this point, which appear sometimes in the form of Porisms, and sometimes of theorems. Of the former, some curious instances will be found in Dr. SMALL's demonstrations of Dr. STEWART's Theorems *. Of the latter, I shall only add one, omitting the demonstration, which would lead into too long a digression.

"If Q be the point in a triangle from which perpendiculars are drawn to the sides of the triangle, so that the sum of their squares is the least possible; twice the area of the triangle is a mean proportional between the sum of the squares of the sides of the triangle, and the sum of the squares of the above-mentioned perpendiculars."

29. But to return to the subject of Porisms: It is evident from what has now appeared, that in some instances at least, there is a close connection between these propositions and the *maxima* or *minima*, and, of consequence the impossible cases of problems. The nature of this connection requires to be further investigated, and is the more interesting, that the transition from the indefinite, to the impossible cases of a problem seems to be made with wonderful rapidity. Thus, in the first proposition, though there be not, properly speaking, an impossible case, but only one where the point to be found goes off *ad infinitum*, we may remark, that if the given point F be any where out of the line HD, the problem of drawing GB equal to GF is always possible, and admits just of one solution; but if F be in the line DH, the problem admits of no solution at all, the point being then at an infinite distance, and therefore impossible to be assigned. There is however this exception, that if the given point be at K, in the same line DH, determined by making DK equal DL, then every point in the line DE gives a

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* Trans. R. S. Edin. vol. ii. p. 112, &c.

solution, and may be taken for the point G. Here therefore the case of innumerable solutions, and the case of no solution, are as it were *conterminal*, and so close to one another, that if the given point be at K, the problem is indefinite, but that if it remove ever so little from K, remaining at the same time in the line DH, the problem cannot be resolved.

I had observed this remarkable affinity, between cases, which in other respects are diametrically opposite, in a great variety of instances, before I perceived the reason of it, and found, that by attending to the origin which has been assigned to Porisms, I ought to have discovered it *a priori*.

30. It is supposed above, that *two* of the conditions of a problem involve in them a third, and wherever that happens, the conclusion which has been deduced will invariably take place. But a Porism may sometimes be so simple, as to arise from the mere coincidence of *one* condition of a problem with another, though in no case whatever, any inconsistency can take place between them. Thus, in the second of the foregoing propositions, the coincidence of the point given in the problem with another point, *viz.* the centre of gravity of the given triangle, renders the problem indeterminate; but as there is no relation of distance, or position, between these points, that may not exist, so the problem has no impossible case belonging to it. There are, however, comparatively but few Porisms so simple in their origin as this, or that arise from problems in which the conditions are so little complicated; for it usually happens, that a problem which can become indefinite, may also become impossible; and if so, the connection between these cases, which has been already explained, never fails to take place.

31. Another species of impossibility may frequently arise from the porismatic case of a problem, which will very much

affect the application of geometry to astronomy, or any of the sciences of experiment, or observation. For when a problem is to be resolved by help of data furnished by experiment or observation, the first thing to be considered is, whether the data so obtained, be sufficient for determining the thing sought; and in this a very erroneous judgment may be found, if we rest satisfied with a general view of the subject. For though the problem may in general be resolved from the data that we are provided with, yet these data may be so related to one another in the case before us, that the problem will become indeterminate, and instead of one solution, will admit of an infinite number.

Suppose, for instance, that it were required to determine the position of a point F, (*Fig.* 25, *Pl.* v.) from knowing that it was situated in the circumference of a given circle ABC, and also from knowing the ratio of its distances from two given points E and D; it is certain, that in general these data would be sufficient for determing the situation of F: But nevertheless, if E and D should be so situated, that they were in the same straight line with the centre of the given circle; and if the rectangle under their distances from that centre, were also equal to the square of the radius of the circle, then, as was shewn above, (§ 12.) the position of F could not be determined.

This particular instance may not indeed occur in any of the practical applications of geometry; but there is one of the same kind which has actually occurred in astronomy. And as the history of it is not a little singular, affording besides an excellent illustration of the nature of Porisms, I hope to be excused for entering into the following detail concerning it.

32. Sir Isaac Newton having demonstrated, that the trajectory of a comet is a parabola, reduced the actual determination of the orbit of any particular comet, to the solution of a geometrical problem, depending on the properties of the para-

bola, but of such considerable difficulty, that it is necessary to take the assistance of a more elementary problem, in order to find, at least nearly, the distance of the comet from the earth, at the times when it was observed. The expedient for this purpose, suggested by Newton himself, was to consider a small part of the comet's path as rectilineal, and described with an uniform motion, so that four observations of the comet being made at moderate intervals of time from one another, four straight lines would be determined, *viz.* the four lines joining the places of the earth and the comet, at the times of the observation, across which if a straight line were drawn, so as to be cut by them into three points, in the same ratios with the intervals of time above-mentioned; the line so drawn would nearly represent the comet's path, and by its intersection with the given lines, would determine, at least nearly, the distances of the comet from the earth, at the times of observation.

The geometrical problem here employed, of drawing a line to be divided by four other lines given in position, into parts having given ratios to one another, had been already resolved by Dr. Wallis and Sir Christopher Wren, and to their solutions Sir Isaac Newton added three others of his own, in different parts of his works. Yet none of all these geometers observed that peculiarity in the problem which rendered it inapplicable to astronomy. This was first done by M. Boscovich, but not till after many trials, when, on its application to the motion of comets, it had never led to any ſatisfactory result. The errors it produced in some instances were so considerable, that Zanotti, seeking to determine by it the orbit of the comet of 1739, found, that his construction threw the comet on the side of the sun opposite to that on which he had actually observed it. This gave occasion to Boscovich, some years afterwards, to examine the different cases of the problem, and to remark that, in one of them, it became indeterminate; and that, by a curious coincidence, this happened in the only case which could be supposed applicable to the astronomical problem above mentioned; in other words, he found, that in the state of the data, which must there always take place, innumerable lines might be drawn, that would be all cut in the same ratio, by the four lines given in position. This he demonstrated in a dissertation published at Rome in 1749, and since that time in the third volume of his *Opuscula*, A demonstration of it, by the same author, is also inserted at the end of CASTILLON's Commentary on the *Arithmetica Universalis*, where it is deduced from a construction of the general problem, given by Mr. THOMAS SIMPSON, at the end of his Elements of Geometry.[*] The proposition, in BOSCOVICH's words, is this: "Problema quo quæritur recta linea quæ quatuor rectas positione datas ita secet, ut tria ejus segmenta sint invium in ratione data, evadit aliquando indeterminatum, ita ut per quodvis punctum cujusvis ex us quatuor rectis duci possit recta linea, quae ei conditoni faciat satis.[†]"

It is needless, I believe, to remark, that the proposition thus enunciated is a Porism, and that it was discovered by BOSCOVICH, in the same way, in which I have supposed Porisms to have been first discovered by the geometers of antiquity. I shall add here a new analysis of it, conducted according to the method of the preceding examples, and to which the following *lemma* is subservient.

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[*] Elements, p. 243, edit. 3. SIMPSON's solution is remarkably elegant, but no mention is made in it, of the indeterminate case.

[†] Jos. BOSCOVICH Opera, Bassani. tom 3. p. 331.

LEMMA I. *Fig.* 31, *Pl.* vi.

32. If two straight lines, AE and BE, be cut by three other straight lines, AB, CD and EF given in position, and not all parallel to one another, into segments having the same given ratio, they will intercept between them segments of the lines given in position, *viz.* AB, CD, EF, which will also have given ratios to one another*.

* Demonstration.—Through C and E draw CH and EG, both parallel to AB, and let them meet BG, parallel to AE, in H and G. Let GF and HD be joined; and because AC is to CE, that is, BH to HG as BD to DF, by hypothesis, DH is parallel to GF, and has also a given ratio to it, *viz.* the ratio of GB to BH, or of EA to AC. Take GK equal to HD, and join EK, and the triangle EGK will be equal to the triangle CHD, and therefore the angle KEG is given, and likewise the angle KEF; and since the ratio of GK to KF is given, if from K there be drawn KL parallel to EG, meeting EF in L, the ratio of EL to LF will be given. But the ratio of EL to LK is given, because the triangle ELK is given in species; therefore the ratio of FL to FK is given; and the angle FLK being alſo given, the triangle FKL is given in species, as also the triangle FGE. The angle FGE being therefore given, the triangle KGE is given in species, and EG has therefore given ratios to EK and EF. But EG is equal to AB, and EK to CD, therefore AB, CD and EF have given ratios to one another. *Q. E. D.*

Hence to find the ratios of AB, CD and EF; in EF take any part EL, and make as AC is to CE, so EL to LF; through L draw LK parallel to EG or AB, meeting EK, drawn through E parallel to CD in K. then if FK be drawn meeting EG in G, the ratios required are the same with the ratios of the lines EG, EK, EF. This is evident from the preceding investigation.

If it be required to find the position of the line AE, drawn through A, so as to be cut by CD and EF in a given ratio; draw A*c*, any how, cutting DC in *c*, and produce A*c* to *e*, so that A*c* may be to *ce* in the ratio which AC is to have to CE; let *e*E be drawn parallel to DC, intersecting FE in E, and if AE be joined, it is the line required.

different cases of the problem, and to remark that, in one of them, it became indeterminate; and that, by a curious coincidence, this happened in the only case which could be supposed applicable to the astronomical problem above mentioned; in other words, he found, that in the state of the data, which must there always take place, innumerable lines might be drawn, that would be all cut in the same ratio, by the four lines given in position. This he demonstrated in a dissertation published at Rome in 1749, and since that time in the third volume of his *Opuscula*. A demonstration of it, by the same author, is also inserted at the end of CASTILLON's Commentary on the *Arithmetica Universalis*, where it is deduced from a construction of the general problem, given by Mr. THOMAS SIMPSON, at the end of his Elements of Geometry.* The proposition, in BOSCOVICH's words, is this: "Problema quo quæritur recta linea quæ quatuor rectas positione datas ita secet, ut tria ejus segmenta sint invium in ratione data, evadit aliquando indeterminatum, ita ut per quodvis punctum cujusvis ex us quatuor rectis duci possit recta linea, quae ei conditoni faciat satis.†"

It is needless, I believe, to remark, that the proposition thus enunciated is a Porism, and that it was discovered by BOSCOVICH, in the same way, in which I have supposed Porisms to have been first discovered by the geometers of antiquity. I shall add here a new analysis of it, conducted according to the method of the preceding examples, and to which the following *lemma* is subservient.

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* Elements, p. 243, edit. 3. SIMPSON's solution is remarkably elegant, but no mention is made in it, of the indeterminate case.

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LEMMA I. *Fig.* 31, *Pl.* vi.

32. If two straight lines, AE and BE, be cut by three other straight lines, AB, CD and EF given in position, and not all parallel to one another, into segments having the same given ratio, they will intercept between them segments of the lines given in position, *viz.* AB, CD, EF, which will also have given ratios to one another *.

* DEMONSTRATION.—Through C and E draw CH and EG, both parallel to AB, and let them meet BG, parallel to AE, in H and G. Let GF and HD be joined; and because AC is to CE, that is, BH to HG as BD to DF, by hypothesis, DH is parallel to GF, and has also a given ratio to it, *viz.* the ratio of GB to BH, or of EA to AC. Take GK equal to HD, and join EK, and the triangle EGK will be equal to the triangle CHD, and therefore the angle KEG is given, and likewise the angle KEF; and since the ratio of GK to KF is given, if from K there be drawn KL parallel to EG, meeting EF in L, the ratio of EL to LF will be given. But the ratio of EL to LK is given, because the triangle ELK is given in species; therefore the ratio of FL to FK is given; and the angle FLK being alſo given, the triangle FKL is given in species, as alſo the triangle FGE. The angle FGE being therefore given, the triangle KGE is given in species, and EG has therefore given ratios to EK and EF. But EG is equal to AB, and EK to CD, therefore AB, CD and EF have given ratios to one another. *Q. E. D.*

Hence to find the ratios of AB, CD and EF; in EF take any part EL, and make as AC is to CE, so EL to LF; through L draw LK parallel to EG or AB, meeting EK, drawn through E parallel to CD in K. then if FK be drawn meeting EG in G, the ratios required are the same with the ratios of the lines EG, EK, EF. This is evident from the preceding investigation.

If it be required to find the position of the line AE, drawn through A, so as to be cut by CD and EF in a given ratio; draw A*c*, any how, cutting DC in *c*, and produce A*c* to *e*, so that A*c* may be to *ce* in the ratio which AC is to have to CE; let *e*E be drawn parallel to DC, intersecting FE in E, and if AE be joined, it is the line required.

PROP. VI. PORISM. *Fig.* 32, *Pl.* vi.

33. Three straight lines being given in position, a fourth line, also given in position, may be found, such, that through any point whatever a straight line may be drawn, which will intersect these four lines, and will be divided by them into three segments, having given ratios to one another.

Let AB, CD, EF be the three lines given in position, and OL the line to be found, and $\alpha\delta$ a given line, of which the segments $\alpha\beta$, $\beta\gamma$, $\gamma\delta$ have given ratios to one another.

Let A be a given point in the line AB, and suppose, that AO is drawn from it, intersecting the lines CD, EF and OL in the points C, E and O, and divided at these points into the segments AC, CE, EO, having the same ratios to one another, with the given segments $\alpha\beta$, $\beta\gamma$, $\gamma\delta$ of the line $\alpha\delta$. Then, because the lines CD, EF are given in position, and also the point A, the line AE is given in position and magnitude, (§ 32.) and therefore also EO, which has a given ratio to AE; the point O is therefore given.

Again, let B be any point whatever in AB, and let BL be drawn, according to the hypothesis of the Porism, so as to be divided in the points D, F and L, where it intersects the lines CD, EF and OL into the parts, BD, DF and FL, having the same ratios with the parts $\alpha\beta$, $\beta\gamma$, $\gamma\delta$.

Hence the converse of the lemma is easily demonstrated, *viz.* that if AE and BF be two lines that are cut proportionally by the three lines AB, CD, EF; and if AB and EF, the parts of any two of these last, intercepted between AE and BF, be also cut proportionally, any how, in b and f, and if bf be joined, meeting the third line in d, bf will be cut in the same proportion with AE or BF. For if not, let bf' be drawn from b, meeting CD in d', and EF in f', so that $bd' : d'f' :: AC : CE$; then by the lemma, $ab : AB :: Ef' : EF$; and by ſupposition, $ab : AB :: Ef : EF$; therefore $Ef' = Ef$, which is impossible, Therefore, &c.

Let also BG be drawn equal and parallel to AE, and let EG be joined; EG will therefore be parallel to AB, and will be given in position; and if GF be drawn, it will make given angles with EG and EF, because, by the preceding *lemma*, the ratio of AB to EF, that is, of EG to EF is given. Through L draw LN parallel to BG, meeting GF produced in N.

Then because the triangles BFG, LFN are similar, GF is to FN as BF to FL, that is, in a given ratio; and therefore, since FG also makes given angles with the two straight lines EG and EF, given in position, the point N is in a straight line, given in position, and passing through E, *viz.* EN.

Now, since BF is to FL as BG to LN, and also as AE or BG to EO, LN and EO are equal, and being also parallel, OL is parallel to EN, that is, to a line given in position; and the point O, in OL, is given, therefore OL is given in position; which was to be found.

Construction. From any two given points, A and B, in the line AB, draw AE and B′F′ intersecting CD and EF in C, E, D′ and F′, so that AC may be to CE, and B′D′ to D′F′ in the same given ratio of $\alpha\beta$ to $\beta\gamma$, (§ 32.) Produce also AE to O, and B′F′ to L′, so that AE may be to EO, and B′F′ to F′L′ in the same given ratio of $\alpha\gamma$ to $\gamma\delta$. If OL′ be joined, it will be the line required.

For let B′ be any point whatsoever in AB, and as AB′ to AB, so let OL′ be to OL, and let BL be drawn, cutting CD′, and EF′ in D and F, the line BL is divided in these points, similarly to the given line $\alpha\delta$. For since the two lines AO and B′L′ are divided similarly by the three lines AB′, CD′ and OL′, and since two of these last, AB′ and OL′, are also divided similarly to one another by the three lines AO, B′L′ and BL, BL will be divided in D, in the same ratio wherein B′L′ is divided in D′, or AO in C, (*Lem.* I. *Conv.*). In the same way, BL is divided in F, in the same ratio wherein AO is divided in E; BL is therefore similarly divided to AO, or to $\alpha\delta$, which was to be demonstrated.

34. Hence it is plain, " If two similarly divided lines, as AO and BL, be drawn any how, and if straight lines AB, CD, EF, OL, be drawn through the points of division of these lines, innumerable lines may be placed between the lines AB, CD, EF and OL, which will be divided by them, similarly to the lines AO, and BL." For, by what is here demonstrated, every line which cuts any two of the lines AB, CD, &c. proportionally, will also cut the others, proportionally, and will be cut by them into segments having the same ratio to one another, with the segments of the lines AO and BL.

From this it follows, that the astronomical problem, above mentioned, becomes a Porism, and is indeterminate, in the case when the observations of the comet are not very distant from one another. For on this supposition, the arches described by the earth and by the comet during the time in which the observations are made, will not differ much from two straight lines; and these lines will be divided similarly to one another, because each of them will be divided into parts, proportional to the intervals of time between the observations. The places of the earth, at the times of the observations, may therefore be nearly represented by the points A, C, E and O, in the straight line AO, and those of the comet by the points B, D, F and L, in the straight line BL, these lines AO and BL being divided both into parts having the same ratios. The position of BL therefore is not given, since, by the Porism, it may be any line whatever, which cuts the two lines, AB and OL, in a certain ratio.

It is also to be remarked, that in order to render this, or any other geometrical problem, of no use in questions where the data are furnished by observation, and are consequently liable to some inaccuracy, it is not necessary, that the problem should be reduced exactly to the porismatic case; for even on a near approach to that case, a very small error in the data will produce so great an error in the conclusion, that no dependence can be had upon its accuracy.

This will be made evident in the present instance, by considering how the construction of the Porism is subservient to the solution of the other cases of the problem. Suppose that four lines, AB, CD, EF, RS, (*Fig.* 32, *Pl.* vi.) are given in position, and that it is required to draw a straight line that shall be divided by these lines into parts having the ratios of the given lines $\alpha\beta$, $\beta\gamma$, $\gamma\delta$. Let KT be that line, and assuming the points A and B′, and drawing the lines AO, B′L′, so that they may be similarly divided to the line $\alpha\delta$, as in the construction of the Porism, then if OL be joined, it will be given in position, and the extremity K, of the line KT, will be in the line OL, by the Porism; but it is also in the line KS; it is therefore given. Now, by the *lemma*, AT is to TB′ as OK to KL′, and the lines OK and KL′ being given, the ratio of AT to TB′ is given, so that T is given, and therefore TK is given in position. *Q. E. I.*

Now, it is evident, that if RS make a small angle with OL, any error in the determination of that angle will make a great variation in the position of the point K. A small change in it may, for instance, make RS parallel to OL, and consequently will throw off K, to an infinite distance, so that the line, which is sought, will be impossible to be found: and in general, the variation of the position of K, corresponding to a given variation in the angle RKO, will be *cæteris paribus*, inversely as the square of the sine of that angle. The nearer, therefore, that the problem is to the Porism, the less is the solution of it to be depended on, and the more does it partake of the indefinite character of the latter.

35. Sir Isaac Newton has extended the hypothesis of the problem from which the preceding Porism is derived, and has formed from it one more general, which he has also resolved with a view to its application in astronomy. It is this: "To "describe a quadrilateral, given in species, that shall have its "angles upon four straight lines given in position *."

As it is evident, that the former problem is but a particular case of this last, it is natural to expect, that a Porism is also to be derived from it, or that the lines given in position may be such, that the problem will become indeterminate. On attempting the analysis, I have accordingly found this conjecture verified: the investigation depending on a *lemma*, similar to that which is prefixed to the preceding proposition.

LEMMA II. *Fig.* 33, *Pl.* vi.

If two triangles ABC, DEF, similar to a given triangle, be placed with their angles on three straight lines given in position, so that the equal angles in both the triangles may be upon the same straight lines, the ratios of the segments of these straight lines, intercepted between the two triangles, that is, of AD, BE and CF are given †.

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* Prin. Math. lib. i. lem. 27.

† Demonstration.—Complete the parallelogram under AC and AD, *viz.* AG, and on DG describe the triangle DGH, similar and equal to ABC. Join FG, BH and HE. Through G also, draw GK, equal and parallel to HE, and join CK; CK will be equal and parallel to BE, and the triangle CGK equal to the triangle BHE. The angle GCK is therefore given, being equal to the given angle HBE; and the angle GCF being given, the angle FCK is also given.

The triangles DHE, DGF are similar; for the angles FDE, GDH being equal, the angles FDG, EDH are likewise equal; and also, by supposition, FD being to DE as GD to DH, FD is to DG as DE to DH. The angle FGD is therefore equal to the angle EHD, and FG is also to EH, or to KG, as FD to DE, or as GD to DH.

PROP. VII. PORISM. *Fig.* 34, *Pl.* vi.

36. Three straight lines being given in position, a fourth may be found, which will also be given in position, and will be such, that innumerable quadrilaterals, similar to the same given quadrilateral, may be described, having their angles placed, in the same order, on the four straight lines given in position.

Let AD, BE, CF be the three straight lines given in position, and *ablc* a given quadrilateral. Let A be a given point in the

But if GL be drawn parallel to HD, the angle KGL will be equal to the angle EHD, that is, to the angle FGD, and therefore the angle KGF to the angle LGD or GDH ; and it has been shewn, that FG is to GK as GD to DH ; therefore the triangle FGK is similar to the triangle GDH, and is given in species.

Draw GM perpendicular to CF, and GN making the angle MGN equal to the angle FGK or GDH, and let GM be to GN in the given ratio of FG to GK, or of GD to DH. Join CN and NK. Then, because MG : GN :: FG : GK ; MG : FG :: GN : GK ; and the angle MGF being equal NGK, the triangles MGF, NGK are similar, and therefore GNK is a right angle. But since the ratio of MG to GN is given, and also of MG to GC, the triangle GCM being given in species, the ratio of GC to GN is given, and CGN being also a given angle, because each of the angles CGM, MGN is given, the triangle CGN is given in species, and consequently the ratio of CG to CN is given. The angle NCK is therefore given ; and the angle CNK is likewise given, each of the angles CNG, GNK being given, therefore the triangle CNK is also given in species. The ratio of CN to CK is therefore given, and since the ratio of CN to CG is also given, the ratio of CG to CK is given, and the triangle CGK given in species. The angle KGC is therefore given, and the angle KGF being also given, the angle CGF is given, and consequently the ratio of CG to CF. The ratios of the lines CG, CK and CF to one another, that is, of AD, BE and CF to one another, are therefore given. *Q. E. D.*

Cor. Hence also it appears, how a triangle given in species may be described, having its angles on three straight lines given in position, and one of the angles at a given point in one of the lines. The solution of this problem is therefore taken for granted, in the analysis of the Porism, though, for the sake of brevity, the construction is omitted.

line AD, and let ABLC be a quadrilateral, similar to the given quadrilateral *ablc*, placed, so that the angles of the triangle ABC, similar to the given triangle *abc*, may be, one of them, at the given point A, and the other two, on the lines BE and CF. The points B and C, and the triangle ABC, will therefore be given, (*Lemma* 2. *Cor.*) and consequently the triangle CBL will also be given in position and magnitude, and the point L will be given. The line to be found must pass through L; let it be LM; let M be any point in it whatsoever, and let MEDF be a quadrilateral similar to the given quadrilateral *ablc*, having its angles on the four lines LM, CF, BE and AD, the angle at M being equal to the angle CLB, &c.

Complete the parallelogram AG, under CA, AD, and on DG describe the quadrilateral GDHN, similar and equal to the quadrilateral ABLC; join BH and LN, and it is evident, that the three lines CG, BH and LN are all equal, and parallel to AD, and are all given in position. Join also AL, DN, DM, MN and FG,

Because the two quadrilaterals DEMF, DHNG are similar the angle FDM is equal to the angle GDN, and therefore the angles GDF to the angle NDM. For the same reason also, GD : DF : : ND : DM, and therefore the triangles GDF, NDM are similar, and the angle FGD equal to the angle MND, and FG : MN : : GD : DN, so that FG has a given ratio to MN.

But because the triangles ABC, DEF are similar, CG has a given ratio to CF, (*Lem.* 2.) so that the angle GCF being given, the triangle CGF is given in species, and FG has to GC a given ratio; now, FG was also shewn to have to MN a given ratio; therefore MN has a given ratio to CG, that is, to LN.

Again, since the triangle CGF is given in species, the angle CGF is given, and CGD being also a given angle, the angle FGD is given, and therefore MND, which is equal to it. But

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